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ON THE NOTION AND FUNCTION OF THE REJECTION OF PROPOSITIONS

Abstract. The paper dwells upon the genesis, development and generalization of the notion of the rejection of propositions. It also outlines the more important results of the methodological research on the concepts of rejection and decidability of deductive systems as understood by Jan Łukasiewicz and his disciple Jerzy Słupecki, the pioneers of these studies.

1. THE GENESIS OF THE NOTION OF REJECTION

The notion of rejection was introduced into formal logic by Łukasiewicz in his work *Logika dwuwartościowa* (*Two-valued logic*) [1921], in which, apart from the term "assertion" (introduced by Frege), Łukasiewicz introduces also the term "rejection". In adding "rejection" to "assertion" he, as he states himself, followed Brentano, but he did not mention anymore about it. As Łukasiewicz writes, he does not define these terms, explaining only that:

> when speaking or writing "I assert the proposition p" or "I reject the proposition r" I mean that I assert or reject the object denoted by the proposition p or the proposition r.

He never again used the method of construction of a system of propositional calculus for two-valued logic in any other systems of propositional calculi. However, the notion of rejected proposition later played an important role in his research on Aristotle's syllogistic [1939], [1951], as well as in his metalogical studies of some propositional calculi [1952], [1953].

In that research Łukasiewicz makes use of the idea of rejection originated by Aristotle. Łukasiewicz applied it to complete syntactical characterization of deductive systems using an axiomatic method of rejection introduced by him into the formal logic in his paper on Aristotle's syllogistic [1939], and then, in a monograph [1951], which were the results of his many years' research on Aristotle's logic, and which included a detailed elaboration of the results presented in the work prepared before the war. As was pointed out by Łukasiewicz (see [1951], p. 67),

> Aristotle, in his systematic investigations of syllogistic forms, not only proves the true ones but also shows that all the others are false, and must be rejected.

Further, on p. 74, Łukasiewicz observes:

Aristotle rejects invalid forms by exemplification through concrete terms. This procedure is logically correct, but it introduces into the systems terms and propositions not germane to it. There are, however, cases where he applies a more logical procedure, reducing one invalid form to another already rejected. On the basis of this remark, a rule of rejection could be stated corresponding to the rule of detachment by assertion; this can be regarded as the commencement of new field of logical inquiries and of new problems that have to be solved.

Moreover, Łukasiewicz [1951], p. 71, writes:

Modern formal logic, as far as I know, does not use *'rejection'* as an operation opposed to Frege's *'assertion'*. The rules of rejection are not yet known.

As a rule of rejection corresponding to the rule of detachment by assertion, Łukasiewicz adopts [1939], [1951] the following rule, which was anticipated by Aristotle:

r_1^-: **the rule of rejection by detachment**: if the implication 'If α, then β' is asserted, but its consequent β is rejected, then its antecedent α must be rejected, too.

As a rule of rejection corresponding to the rule of substitution for assertion, Łukasiewicz adopts [1939], [1951] the following rule which was unknown to Aristotle:

r_2^-: **the rule of rejection by substitution**: if β is a substitution instance of α, and β is rejected, then α must be rejected, too.

Both rules enable us to reject some syllogistic forms, provided that some other forms have already been rejected. As we mentioned above, Aristotle used the procedure of rejection of some forms by means of concrete terms, but such a procedure, though correct, introduces into logic terms and propositions that are not germane to it.

To avoid this difficulty, Łukasiewicz rejects some forms axiomatically, which leads him to biaspectual axiomatic characterization of deductive systems analyzed by him [1939], [1951], [1952], [1953]. The main idea of such a syntactical characterization consists in providing both:

1° the axioms and inference rules for the given deductive system,

which intuitively lead from some true formulas to true ones of this system,

2° the rejected axioms (treated as false formulas of this system) and rejection (refutation) rules of this system,

which intuitively lead from some false formulas to false ones of this system.

Next, we will outline the biaspectual axiomatic method of characterization of the deductive system, introduced by Łukasiewicz, and applied by him, for the first time, in formal exposition of the Aristotelian logic [1939], [1951].

2. ŁUKASIEWICZ'S BIASPECTUAL FORMALIZATION OF ARISTOTELIAN LOGIC

Aristotelian logic is a theory of names. As it was demonstrated by Łukasiewicz in his monograph on Aristotle's syllogistic [1951], it was the first logical system — the axiomatic system — in the history of European thought.

With reference to Aristotle's theory of names, Łukasiewicz constructed an axiomatic system of calculus of names, called by him *Aristotle's syllogistic*. He presented it in *Elements of Mathematical Logic* [1929]. Characterizing this system we do not use Łukasiewicz's parenthesis-free notation but the one that is currently in use. We denote the system by *AS*.

The system *AS* is based on the classical propositional calculus CL, whose axiomatic dates 1924 and was published in 1925 (see Łukasiewicz [1925]).

VOCABULARY OF *AS*:

- constant symbols of CL, i.e., the connectives of CL,
- primitive terms of *AS*: constants *a* and *i*, which are sentence forming functors of two-term arguments: 'all… are…' and 'some… are…',
- nominal variables: S, P, M, N, …

WELL-FORMED EXPRESSIONS OF *AS*:

- atomic affirmative expressions: formulas of the form S *a* P and S *i* P, which are read: 'all S are P' and 'some S are P', respectively,
- compound expressions: formulas that are created from the atomic ones using the connectives of CL.
- the set **F** of all well-formed expressions — the smallest set of formulas including the atomic expressions and closed under the connectives of CL.

Let us note that, to the set F, there belong not only atomic expressions and all syllogistic moods but also formulas that are not syllogistic moods of Aristotelian logic.

DEFINED TERMS OF *AS*:

- the remaining constants of Aristotelian logic: *e* and *o*, i.e. the functors: 'no… are…' and 'some… are not…' which are defined as follows:

D1. $\mathrm{S}\ e\ \mathrm{P} \underset{\mathrm{df}}{=} \sim \mathrm{S}\ i\ \mathrm{P}$,

D2. $\mathrm{S}\ o\ \mathrm{P} \underset{\mathrm{df}}{=} \sim \mathrm{S}\ a\ \mathrm{P}$.

The negative expressions S *e* P and S *o* P are read: 'no S are P' and 'some S are not P', respectively. Apart from the atomic expressions, we count them among so-called *simple expressions*.

AXIOMS OF *AS*:

$A^{+}1.$ S *a* S,

$A^{+}2.$ S *i* S,

A^+3. M a P ∧ S a M → S a P (B a r b a r a),

A^+4. M a P ∧ M i S → S i P (D a t i s i).

Axioms A^+1 and A^+2 are the two laws of identity; Aristotle did not accept them.

PRIMITIVE INFERENCE RULES FOR *AS*:

r_0^+: the rule of definitional replacement
(according to D1, S e P may be everywhere replaced by ∼ S i P, and, according to D2, S o P may be everywhere replaced by ∼ S a P);

r_1^+: the rule of detachment (*modus ponens*)
(if '$\alpha \to \beta$' and α are asserted expressions of a system, then β is an asserted expression);

r_2^+: the rule of substitution
(if α is an asserted expression of the system, then any expression produced from α by a valid substitution is also an asserted expression; the valid substitution is to put, for term-variables, other term-variables).

The schemes of the rules r_1^+ and r_2^+ are as follows:

$$r_1^+: \frac{\vdash \alpha \to \beta \quad \vdash \alpha}{\vdash \beta} \qquad r_2^+: \frac{\vdash \alpha}{\vdash e(\alpha)}$$

The symbol '⊢' is a sign of assertion introduced by Frege, whereas the expression '$e(\alpha)$' denotes a substitution instance of α.

Characterization of the system *AS* on the second level consists in supplementing it with rejected axioms and rejection rules. Łukasiewicz formulates the following rejected axioms and rejection rules:

REJECTED AXIOMS OF *AS*:

A^-1. P a M ∧ S a M → S i P,

A^-2. P e M ∧ S e M → S i P.

PRIMITIVE REJECTION RULES FOR *AS*:

r_1^-: the rule of rejection by detachment (reverse of *modus ponens*),

r_2^-: the rule of rejection by substitution.

The schemes of the rules r_1^- and r_2^- are the following:

$$r_1^-: \frac{\vdash \alpha \to \beta \quad \dashv \beta}{\dashv \alpha} \qquad r_2^-: \frac{\dashv e(\alpha)}{\dashv \alpha}$$

The symbol '⊣' is the sign of rejection.

The characterized system *AS* is determined by the following ordered 5-tuple, which may be called the *basis* of *AS*:

(B) $\langle F, A^+, R^+; A^-, R^- \rangle$,

where F is the set of all well-formed formulas of this systems, A^+ the set of its axioms, R^+ the set of its primitive inference rules, A^- the set of its rejected axioms, and R^- the set of its primitive rejection rules. The tuples

$$\langle F, A^+, R^+ \rangle \quad \text{and} \quad \langle F, A^-, R^- \rangle$$

determine, respectively, the set T^+ of all *theses* of this system and the set T^- of all its *rejected formulas*.

The first tuple may be called the *asserted system* for *AS*, whereas the second one may be called the *refutation system* for *AS*. T^+ is the set of all well-formed formulas derivable from the set of theses of metalogically formulated CL and axioms of A^+ by means of inference rules of R^+, i.e.,

$$T^+ = Cn^+(CL \cup A^+, R^+),$$

and T^+ is the smallest set including $CL \cup A^+$ and closed under the inference rules of R^+. T^- is the set of all well-formed formulas derivable from the rejected axioms A^- by means of T^+ and rejection (refutation) rules of R^-, i.e.,

D(Ł). $T^- = Cn^-(T^+ \cup A^-, R^-)$,

and T^- is the smallest set including A^- and closed under the rejection rules of R^-.

The set T^- is the *set of all rejected expressions of the system in Łukasiewicz's sense*. To the set T^+ there also belong all 24 valid syllogistic forms, the laws of logical square and the laws of conversion, and to the set T^- of all rejected formulas there belong all the remaining 232 invalid forms. However, it turned out that there exists such a well-formed expression of *AS* which is neither a thesis of this system nor a rejected expression of the set T^-. Such, for example, is the formula:

(F1) S i P → (∼ S a P ∧ P a S).

In order to remove this difficulty, we could reject the expression (F1) axiomatically. However, a question arises whether there exists some other formula of the same kind as (F1), or, may be, an infinite number of such formulas, which can be called *undecidable* on the strength of our basis (B). Therefore, we may only claim that the following condition holds:

$$T^+ \cup T^- \subset F$$

and that the system *AS* whose basis is (B), analyzed by us, is not *saturated* or *decidable* in the sense that it is both 1° *Ł-consistent* and 2° *Ł-complete*, i.e.

$$1°\ T^+ \cap T^- = \emptyset \quad \text{and} \quad 2°\ T^+ \cup T^- = F.$$

And a system satisfying both conditions was later called by Słupecki an Ł-*decidable system* (see Słupecki, Bryll, Wybraniec-Skardowska [1971]). Łukasiewicz does not provide definitions of the terms "consistent system", "complete system", and "decidable system", used by himself, but the contexts alone imply the meaning that these terms were given by Słupecki. The problem was solved by Słupecki providing a basis for which the system of Aristotelian syllogistic is Ł-decidable (see Łukasiewicz [1939], [1951]).

3. SŁUPECKI'S SOLUTION OF THE PROBLEM OF Ł-DECIDABILITY OF ARISTOTELIAN SYLLOGISTIC; STRENGTHENING OF THE NOTION OF THE REJECTED PROPOSITION

The problem concerning the finite Ł-decidability of Aristotelian syllogistic was raised by Łukasiewicz in December 1937, during his seminar on Mathematical Logic at the University of Warsaw. Łukasiewicz presented the problem in the form of the following questions:

Q1. Are the axioms of A^+ for *AS* together with the inference rules of R^+ for *AS* sufficient to prove all true expressions of the *AS*?

Q2. Are the rules of rejection of $R^- = \{r_1^-, r_2^-\}$ for *AS* sufficient to reject all false expressions (every formula of F that is not a thesis of T^+), provided that a finite number of them are rejected axiomatically?

Jerzy Słupecki, who participated in Łukasiewicz's seminar, found solutions to both problems as early as the beginning of 1938 and presented them at one of the seminar meetings. His answer to the question Q1 was positive, to the second – negative. Słupecki was able to prove that it is not possible to reject all the false expressions of *AS* by means of the rules r_1^- and r_2^-, provided a finite number of them is rejected axiomatically:

$$\forall A^- \subseteq F\,[\mathrm{card}(A^-) < \aleph_0 \Rightarrow \exists \alpha \in F \setminus T^+\ (\alpha \notin Cn^-(CL \cup A^-, R^-))].$$

More precisely: Słupecki proves that no matter how many false expressions we may reject axiomatically there always exists some other false expression that does not belong to T^+ and which cannot be rejected otherwise than axiomatically. This way Słupecki gave a negative answer to the question Q2:

(i) The system *AS* within the basis (B) is not Ł-complete with any finite set of axioms A^-.

The negative answer to the question Q2 left without an answer the question Q1. Since it is impossible to establish an infinite set of A^- of rejected axioms, a new rejection rule must be added to the set $R^- = \{r_1^-, r_2^-\}$ of original rules used by Łukasiewicz (called in the literature *Łukasiewicz's rejection rules*). Słupecki extended the system *AS*, adding to it a new rejection rule called by Łukasiewicz *Słupecki's rule of rejection.* It is denoted by r_S^- and has the following scheme:

$$r_S^-:\ \frac{\dashv \alpha \Rightarrow \gamma \qquad \dashv \beta \Rightarrow \gamma}{\dashv \alpha \wedge \beta \Rightarrow \gamma}$$

where α and β denote negative expressions in the form: S *e* P or S *o* P and γ denotes a simple expression or an implication the consequent of which is a simple expression and the antecedent is a conjunction of such expressions. Słupecki's rule says: "If the expression γ does not follow from any of two negative expressions then it does not follow from their conjunction." Słupecki's rule is closely related to the principle of traditional logic (*ex mere negative nihil sequitur*).

As was noticed by Łukasiewicz, having added Słupecki's rule, it is enough to adopt merely one rejected axiom, namely, A^-1. Słupecki demonstrated that:

(ii) System of Aristotle's syllogistic determined in the following base:

(B) $$\langle F, A^+, R^+, \{A^-1\}, R^- \cup \{r_S^-\}\rangle$$

is an Ł-decidability system, i.e.

$$1^\circ\ T^+ \cap \hat{T}^- = \emptyset, \qquad 2^\circ\ T^+ \cup \hat{T}^- = F,$$

where

D(Ł̂) $$\hat{T}^- = Cn^-(T^+ \cup \{A^-1\}, R^- \cup \{r_S^-\})$$

and $\hat{T}^-$ is the *set of all rejected propositions*, i.e. the set of propositions derivable from the axiom A^-1 by means of the thesis of *AS* and Łukasiewicz's rejection rules and Słupecki's rejection rule.

It is then clear that Słupecki extended the notion of the rejected proposition (see (ii), D(Ł̂)) used by Łukasiewicz, because:

D(Ł) ⊂ D(Ł̂). $$T^- \subset \hat{T}^-.$$

In the proof of the condition 2°, i.e. of Ł-completeness, Słupecki used a syntactic method of transforming expressions into a special form in which they are either theses or rejected expressions. In the proof of the condition 1°, Słupecki made use of the condition 2° as well as the arithmetical interpretation of syllogistic given by Leibniz; according to this interpretation, the axiom A^-1 is false, and all adopted rules of rejection lead from false propositions to false propositions.

In this way, the problem of Ł-decidability finds its solution: any well-formed formula of the system of Aristotle's syllogistic is either a thesis (it is true) or a rejected formula (it is false) and the refutation system

(iii) $\langle F, \{A^-1\}, R^- \cup \{r_S\}\rangle$

is a *complete refutation system* of Aristotle's syllogistic. It is obvious that by this method Słupecki managed to give a positive answer to the question Q1.

The results obtained by Słupecki were summarized by Łukasiewicz in his work [1939] **containing also the text of his paper on Aristotle's syllogistic presented by him at the Meeting of the Polish** Academy of Sciences in Cracow on June 6 of 1939. **The work was published** after the war, but was dated 1939. The results of research of both Łukasiewicz and Słupecki were later presented in detail in Łukasiewicz's monograph [1951]. In both **works, Łukasiewicz expressed his high opinion of** Słupecki's findings, which, in **the words of Łukasiewicz** [1939], **being "organically"** united with researches of **the author** ... the author regards **as the most** significant discovery made **in the field of syllogistic since Aristotle."**

Słupecki failed to publish his findings before the war. Similarly to the case of **Łukasiewicz's monograph, his work, prepared before the war,** and containing **the results of his research, was burned during the war. Again,** similarly to Łukasiewicz, Słupecki reconstructed his findings after the war ends and published them in a monograph [1948]. The work was published with a number of changes and was presented as a habilitation dissertation. Słupecki dedicated it to his teacher – Jan Łukasiewicz.

In the monograph [1948], in the proof of Ł-consistence (condition (ii), 1°), Słupecki used a simple set-theoretical **interpretation**. In his proof of Ł-completeness (condition (ii), 2°), he used **also a definition** of the rejected proposition different than Łukasiewicz D(Ł), **and, additionally**, he modified its extension D(Ł), adopted earlier by himself. Instead of Łukasiewicz's definition D(Ł), Słupecki adopts the following equivalent definition:

D(Sł) A *rejected proposition* on the ground of the basis

$$(B, R^-) \qquad \langle F, A^+, R^+, A^-, \emptyset \rangle$$

is such an expression for which there exists a rejected axiom of the set A^- which is derivable from it and theses of the set T^+ by means of inference rules R^+.

The definition of a rejected proposition D(Sł) is closer to Aristotle's idea of refutation of **syllogisms by means of** reducing them to syllogisms rejected earlier. Denoting a **set of all rejected propositions** in the sense of the definition D(Sł) by $Cn^{-1}(T^+ \cup A^-, R^-)$ **we obtain** the following symbolic notation of it:

$$D(Sł) \quad \alpha \in Cn^{-1}(T^+ \cup A^-, R^+) \Leftrightarrow \exists \beta \in A^- \ (\beta \in Cn^+(T^+ \cup \{\alpha\}, R^+)),$$

where Cn^+ (as used here and further on) is a consequence operation with respect to the set T^+ of all theses of the system and its set of rules R^+.

We note (see D(Ł)) that:

$$D(Ł) \approx D(Sł). \quad T^- = Cn^-(T^+ \cup A^-, R^-) = Cn^{-1}(T^+ \cup A^-, R^+).$$

Thus, the notions of rejected propositions, both the one used by Łukasiewicz and that introduced by Słupecki in the form of the definition D(Sł), are equivalent. This observation made it possible for Słupecki [1959] to generalize the notion of rejection into a function Cn^{-1} of rejection, called by him *Łukasiewicz's function*, to analyze properties of the function and to formulate foundations of a theory of rejected propositions (see Wybraniec-Skardowska [1969], Bryll [1969] and Słupecki, Bryll, Wybraniec-Skardowska [1971] and [1972]). The generalizations of the notion of rejection and the theories of the rejection of propositions will be discussed briefly in Section 6. Before that, however, we will reconstruct Słupecki's definition of an extended notion of the rejected proposition, equivalent to the definition D(Ł̂). In the definition D(Ŝł) of the extended notion Słupecki used the function Cn^{-1} for any set X of formulas of F. We obtain the definition $D(Sł; A^-/X)$ of such a function by replacement in D(Sł) the term 'A^-' by the term 'X'.

D(Ŝł). A *rejected proposition* on the ground of the basis

$$(\hat{B}) \qquad \langle F, A^+, R^+; \{A^-1\}, \{r_S^-\} \rangle$$

is either a rejected axiom A^-1 or a proposition rejected with respect to those rejected earlier in the sense of $D(Sł; A^-/X)$, or an expression rejected on the basis of those rejected earlier by means of Słupecki's rule r_S^-.

Let $Cn'(T^+ \cup \{A^-1\}, R^+, \{r_S^-\})$ be the set of all rejected propositions in the sense of D(Ŝł).

We may note that

$$D(\hat{Ł}) \approx D(\hat{Sł}).$$

$$\hat{T}^- = Cn^-(T^+ \cup \{A^-1\}, R^- \cup \{r_S^-\}) = Cn'(T^+ \cup \{A^-1\}, R^+; \{r_S^-\}).$$

It is easy to see that the given definitions of rejected propositions provide three different ways of understanding this notion (see Diagram 1).

$$\begin{array}{llcl}
D(Sł) \subset D(\hat{Sł}). & Cn^{-1}(T^+ \cup A^-, R^+) & \subset & Cn'(T^+ \cup \{A^-1\}, R^+; \{r_S^-\}) \\
 & \| & & \| \\
 & T^- & \subset & \hat{T}^- \\
 & \| & & \| \\
D(Ł) \subset D(\hat{Ł}). & Cn^-(T^+ \cup A^-, R^-) & \subset & Cn^-(T^+ \cup \{A^-1\}, R^- \cup \{r_S^-\})
\end{array}$$

Diagram 1

The first way refers to Łukasiewicz's understanding of the rejected proposition (see D(Ł)) and to its strengthening given by Słupecki (see D(Ł̂)); the second and the third methods relate to Słupecki's understanding of the rejected proposition (see D(Sł) and D(Ŝł)). At the same time, the second refers directly to Aristotle's method of rejection of syllogisms by reducing them to previously rejected syllogisms and makes it possible to simplify the procedure of rejection without supplementing the system with the rules of rejection, and the third one (see $D(\hat{S}ł)$) is a combination of both former methods.

The first method can be called *Łukasiewicz's*, the second – *Słupecki's*, and the third *Słupecki–Łukasiewicz's* method. This last method, at the same time, using the specific (provided by Słupecki) rule of rejection for the system *AS* of Aristotle's syllogistic and not using Łukasiewicz's rejection rules makes it possible to obtain every rejected proposition of *AS* in its strengthened form set by $D(Ł)$.

4. THE NOTION OF REJECTED PROPOSITION AND THE NOTION OF DECIDABILITY

In Sections 2 and 3, we presented three ways of understanding the notion of rejected proposition, which indicate the existence of three different methods of rejection and of three different notions of rejected proof on the ground of one of the bases: (B), $(B\backslash R^-)$, $(\hat{B})$ of Aristotle's syllogistic *AS*. In this section, we are going to demonstrate the possibility of adapting the previously adopted definitions for any deductive system and, with reference to it, we will present three different ways of defining the notion of Ł-decidability. Next, we will show the relationship between this notion and the notion of decidability in the usual sense.

Let S be any deductive system with biaspectual formalization and with the basis

(B_S) $$\langle F_S, A_S^+, R_S^+; A_S^-, R_S^- \cup R_S' \rangle$$

determined, respectively, by computable sets: the set F_S of all well-formed formulas, the set A_S^+ of axioms (asserted axioms), the set R_S^+ of inference rules, the set A_S^- of rejected axioms, and by the set $R_S^- \cup R_S'$ of rejection rules, with assumption that R_S^+ and R_S^- are sets of mutual dual rules (the sole premise of any rule r^- of R_S^-, not being a thesis of S, is the conclusion of a rule r^+ of R_S^+ whereas the conclusion of the rule r^- is a premise of the rule r^+) while R_S' is a set of non-dual rejection rules.

Let T_S^+ be a set of all theses of S. Then, according to Łukasiewicz's conception, by analogy with $D(Ł)$, the set ${}^1T_S^-$ of all rejected propositions of S, with respect to the set T_S^+ and the basis (B_S) (the refutation systems $\langle F_S, A_S^-, R_S^- \cup R_S' \rangle$), is defined as follows:

$D_S(Ł)$. $${}^1T_S^- = Cn_S^- (T_S^+ \cup A_S^-, R_S^- \cup R_S')$$

and ${}^1T_S^-$ is a set of all formulas with rejection proofs in Łukasiewicz's sense, i.e. a set of all formulas derivable from rejected axioms of A_S^- by means of theses of T_S^+ and rejection rules of $R_S^- \cup R_S'$, i.e. ${}^1T_S^-$ is the smallest set including A_S^- and closed under the rejection rules of $R_S^- \cup R_S'$.

If $R_S' = \emptyset$, the basis (B_S) of the system S can be replaced by the basis

$(B_S\backslash R_S^-)$. $$\langle F_S, A_S^+, R_S^+; A_S^-, \emptyset \rangle,$$

and the set ${}^2T_S^-$ of all rejected propositions of S with respect to the set T_S^+ and the basis $(B_S\backslash R_S^-)$, by analogy with $D(Sł)$, is defined as follows:

$D_S(Sł)$. $${}^2T_S^- = Cn_S^{-1}(T_S^+ \cup A_S^-, R_S^+)$$

and ${}^2T_S^-$ is a set of all propositions with rejection proofs in Słupecki–Staszek's sense (see Staszek [1971]), i.e. a set of all such formulas from which and from theses of T_S^+ and by means of inference rules of R_S^+ a rejected axiom of A_S^- is derivable.

If $R_S' \neq \emptyset$, the basis (B_S) can be replaced by the basis

$(\hat{B}_S)$ $$\langle F_S, A_S^+, R_S^+; A_S^-, R_S' \rangle$$

and the set ${}^3T_S^-$ of all rejected propositions of S with respect to the set T_S^+ and the basis $(\hat{B}_S)$ (the refutation system $\langle F_S, A_S^-, R_S' \rangle$) by analogy with $D(\hat{S}ł)$ can be defined as follows:

$D_S(\hat{S}ł)$. $${}^3T_S^- = Cn_S'(T_S^+ \cup A_S^-, R_S^+; R_S')$$

and the set ${}^3T_S^-$ is a set rejected propositions with rejected proofs in Słupecki–Łukasiewicz's sense (see Słupecki [1972]), i.e. it is a set of all such propositions every one of which is either: 1° a rejected axiom of A_S^- or 2° a proposition rejected with respect to those rejected earlier in Słupecki–Staszek's sense, or 3° a proposition rejected on the basis of those rejected earlier by means of rejection rules of R_S'.

Let us note that the definitions $D_S(Ł)$, $D_S(Sł)$ and $D_S(\hat{S}ł)$ are not finally precise and require the precise definitions in the above-mentioned rejection proofs on the ground of any set $X \subseteq F_S$ with respect to the bases (B_S), $(B_S\backslash R_S^-)$ and $(\hat{B}_S)$ (see Słupecki [1972], Słupecki and Bryll [1977], Wybraniec-Skardowska [1983]). For the sake of the present work, we have simplified it to some extent.

Among the sets of rejected propositions defined above, the following relationships (similar to those presented in Diagram 1) hold:

FACTS. a. *If* $R_S' = \emptyset$, *then* ${}^2T_S^- = {}^1T_S^-$. b. ${}^2T_S^- \subseteq {}^1T_S^- = {}^3T_S^-$.

The three different definitions of the sets of rejected propositions of the system S entail three different definitions of Ł-decidability of this system.

DEFINITION. The system S is *Ł-decidable* if and only if, for some $i = 1, 2, 3$, S is *^{i}Ł-decidable*, i.e. it satisfies the following two conditions:

$$1°\ T_S^+ \cap {}^iT_S^- = \emptyset, \quad 2°\ T_S^+ \cup {}^iT_S^- = F_S.$$

The condition 1° is called the *^{i}Ł-consistence* condition and the condition 2° is called the *^{i}Ł-completeness* condition.

A question arises: What is the relationship between Ł-decidability and decidability in the usual sense? The answer was given by Słupecki [1972], who founded his considerations on a certain observation of Borkowski (see [1970], p. 331):

SŁUPECKI'S THEOREM. *If the system S is Ł-decidable and any rejection rule of $R_4 \cup R'_5$, except for the rule of rejection by detachment, is computable, then the system S is decidable in the usual meaning.*

Słupecki's theorem is an immediate consequence of the following theorem of the theory of recursion, which we quote from Grzegorczyk's monograph [1974]:

If the union of two recursively enumerable disjoint sets T and S is a computable set, then the sets T and S are also computable.

It is easy to prove that the Ł-decidable system of Aristotle's syllogistic with the basis (B̄) (see Section 3) is decidable. The Ł-decidable systems presented in the next section are also decidable.

5. MORE IMPORTANT FINDINGS CONCERNING Ł-DECIDABILITY OF DEDUCTIVE SYSTEMS

The axiomatic rejection method introduced by Łukasiewicz to complete, biaspectual characterization of deductive systems (which was effectively used and developed by Słupecki) produced a broad response in literature after the Second World War. Łukasiewicz, already living in Dublin, uses the method in his research on intuitionistic logic [1952] as well as in a four-valued modal system of propositional calculus [1953], built by himself. At the same time, in Poland, further studies, inspired by Słupecki on Ł-decidability of deductive systems and the very notion of rejected proposition were taken up.[1] In studies on Ł-decidability and production of complete refutation systems for logical systems (asserted systems) one of three methods, modeled on those described in the previous section, are usually used. In the sequel we will present a few, more important results connected with this research.[2]

5.1. Calculi of names. Słupecki's research on Aristotle's syllogistic was continued mainly by Iwanuś. Using Słupecki–Łukasiewicz's methods, Iwanuś managed to prove Ł-decidability of a few systems of calculi of names. With the help of techniques which Słupecki had used in his proof of Ł-decidability of the system *AS* of Aristotle's syllogistic, Iwanuś [1973] gave a proof of Ł-decidabili-

[1] I provide a detailed presentation of these studies in [1983].

[2] G. Bryll (as well as his book [1996]) and T. Skura have been very helpful in verifying certain significant facts. I would like to express my sincere gratitude for their efforts.

ty of the whole traditional calculus of names, i.e. the system of Aristotle's syllogistic enriched by nominal negation.

Another interesting, though much later obtained, result of Iwanuś's research [1992] is a proof of Ł-decidability of the system of Aristotle's syllogistic built by Słupecki [1946]. In this system, the two initial Łukasiewicz's axioms (laws of identity) of the system *AS* (those that are absent in Aristotle's logic) are replaced with the following axioms:

$$S\ a\ P \Rightarrow S\ i\ P, \qquad S\ i\ P \Rightarrow P\ i\ S.$$

In Słupecki's system, unlike in the system *AS*, it is permissible for variables to represent empty names. A complete refutation system for the system given by Iwanuś [1992] is based on three rejected axioms and one rejected rule that is germane to the traditional calculus of names.

As is well known, Leśniewski built a system of calculus of names, called *ontology*, which is immeasurably richer than syllogistic. Słupecki, in [1955], presented a system of calculus of names built upon metalogically formulated quantificational calculus, which he called *Leśniewski's ontology*, and in which he distinguished a fragment called *elementary ontology*. In elementary ontology it is permissible to interpret both the asserted system *AS* of syllogistic and the asserted system with nominal negation. Moreover, in Leśniewski's elementary ontology there exist also counterparts of the majority of known theses of the system *AS* or of other syllogistic systems — richer than *AS* but in which it is permissible for variables to be substituted only by non-empty names. Iwanuś, in [1972], gave a proof of Ł-decidability of a certain version of the system of elementary ontology. Iwanuś presents a complete refutation system in which he distinguishes two independent rejected axioms (which, according to Leśniewski's standpoint, should not be theses of the ontology) and, germane to this system, a non-Łukasiewicz rule of rejection by omission of the universal quantifier.

It should also be noted that, in all the proofs of Ł-completeness, Iwanuś made use only of syntactical techniques, applying the method of transforming well-formed formulas to special equivalent forms; the proofs of Ł-consistence are obtained by Iwanuś by means of completeness theorems for set-theoretical semantics for these systems.

5.2. Propositional logic. As we have already mentioned, Łukasiewicz used to apply the axiomatic method of rejection also to some systems of propositional calculus.

5.2.1. *Classical logic.* Łukasiewicz mentioned in [1953] that the classical sentential calculus with the inference rules r_1^+ and r_2^+ is Ł-decidable. A complete refutation system for it determines one rejected axiom, namely the sentential variable *p*, and two Łukasiewicz's rules r_1^- and r_2^-.

Let us note that the method of rejection of false formulas (i.e. non-theses) used by Łukasiewicz can be replaced by Słupecki's method, omitting Łukasiewicz's rules, i.e. applying the following principle: If from a formula of propositional calculus and the set of theses it is possible to deduce, according to the rules r_1^+ and r_2^+, the rejected axiom p, then the formula is rejected (see D(S!)).

For the classical first-order calculus rejected axioms and rejected rules were formulated by Skura in [1993]. Skura made use of the 'tableaux' method.

The rejection procedure accompanying the syntactic characterization of systems of propositional logic became a standard among logicians. It would be difficult to enumerate all the results obtained with the help of it, therefore, we will limit our presentation to a brief mention of only intuitionistic logic and extensions, modal logic and Łukasiewicz's logics.

5.2.2. *Intuitionistic logic and extensions.* In his research on intuitionistic propositional calculus, Łukasiewicz [1952] advanced the hypothesis that it is Ł-decidable. Moreover, he supposed that a sole rejected axiom of it is a propositional variable, and that the rejection rules are: r_1^-, r_2^- and one special rule (Gödel rule) which states that the alternative $\alpha \vee \beta$ is rejected whenever so are α and β. Thanks to Kreisel and Putnam [1957], we know that the rules proposed by Łukasiewicz do not suffice to reject all non-theses of the intuitionistic calculus. It can be extended by means of the Kreisel–Putnam formula, so that the new system is closed under these rules and, moreover, there are infinitely many such extensions (cf. Maximowa [1986]). It is also known that there is no finite set of rejected axioms that, together with Łukasiewicz's rules, gives a complete refutation intuitionistic system (cf. Maduch [1973]).

Ł-decidability of the intuitionistic propositional logic was achieved by Scott [1957] who used a countable number of non-structural rejection rules. However, Scott's results, which were presented in *Summaries of Talks* at Cornell University, were inaccessible to Słupecki's circle in Poland.

Independently of Scott's results, the proof of Ł-decidability of the intuitionistic propositional calculus was provided by Dutkiewicz [1989]. In his approach, a complete refutation intuitionistic system is compounded of one, already mentioned, axiom and three rejection rules: Łukasiewicz's rules and a new, original rejection rule which is, in fact, an infinite countable class of rejection rules of a common scheme. In his proof, Dutkiewicz uses the method of rejection modeled on the one applied by Łukasiewicz (see D(Ł)) as well as the method of Beth's semantic tableaux.

Another proof of Ł-decidability for the intuitionistic calculus was given by Skura [1989], [1999], who, while defining a complete intuitionistic refutation system, added to Łukasiewicz's rules a new rule or rather a class of structural rules of rejection, the number of which is infinite (in the proof of Ł-decidability, Jaśkowski's matrix and Heyting's algebras are used). Skura in [1989] proved that in any intermediate logic, stronger than intuitionistic calculus, all these rules cannot hold simultaneously. In [1992], Skura provided a complete refutation system for certain intermediate logics.

5.2.3. *Modal logic.* The first research into a complete syntactic characterization of a modal propositional calculus was undertaken, as was already mentioned, by Łukasiewicz in [1953]: he extends the four-valued modal system, built by himself, by two rejected axioms and rejection rules r_1^- and r_2^- obtaining Ł-decidability of this system. The axioms for this system in his notation are as follows:

$A^+1.$	$CpCNpq$,	$A^-1.$	$C\Delta pp$,
$A^+2.$	$CCNppp$,	$A^-2.$	Δp,
$A^+3.$	$CCpqCCqrCpr$,		
$A^+4.$	$Cp\Delta p$.		

Afterwards, Słupecki initiated research on Ł-decidability of Lewis system S5. In his and Bryll's paper [1973], the proof of Ł-decidability was achieved with an assumption of one rejected axiom (the propositional variable p) and, apart from Łukasiewicz's rules, a class of rejection rules of the common scheme. Skura gave a simpler proof of Ł-decidability of Lewis system S5 assuming that the language of this system was supplemented with a symbol '$\perp$', i.e. the *constant of falsity*. Skura [1992], [1993] adopts this constant as the rejected axiom and extends the systems of Łukasiewicz's rules by 1° a rule stating that: if formula $\Box p$ is rejected, the formula p is rejected, too, and 2° a class of structural rejection rules of the same scheme. In [1992], [1995], [1999], Skura, using the algebraic method, also provided a complete refutation system for the logic S4 and for some of its extension (Grzegorczyk's logic). A little earlier, Goranko [1991], [1994] formulated a complete refutation system for some normal modal propositional logics (including S4 and Grzegorczyk's logic) that are characterized by a class of finite trees. The proof methods that he used are connected with Kripke's semantics. His refutation systems for these logics are based on the same rejected axiom (the constant '$\perp$'), Łukasiewicz's rules and a class of non-structural rejected rules of the same scheme.

The method of constructing refutation systems corresponding to classes of finite models was used by Skura [1992] for intermediate logics, and by Skura [1994] and Goranko [1994] for certain normal modal logics. In [1994], Skura showed that the refutation systems can be useful in such cases when a given system of logic cannot be characterized by any class of finite models (algebras): there is a decidable modal logic without a finite model property that has a simple refutation system.

5.2.4. *Łukasiewicz's many-valued logics.* Researches into Ł-decidability of Łukasiewicz's sentential calculus were conducted in the Opole circle of logical research, which, for many years, was led by Jerzy Słupecki. In [1968], Bryll and Maduch formulated a uniform method of rejection of formulas in an $(n+1)$-

valued implicational, implicative-negative and definitionally complete Łukasiewicz's calculus. In these systems the same formula may be adopted as the sole rejected axiom

$$C(Cp)^n q(Cp)^{n-1} q$$

(for $n = 1$ we get, in particular, the rejected axiom: $CCpqq$ in the classical implicational calculus). The only rejection rules are, here, Łukasiewicz's rules.

A complete refutation system for $\aleph_0$-valued Łukasiewicz's calculus was built by Skura [1993] by extension of Łukasiewicz's rejection rules. Research into Ł-decidability of this system has been conducted also by Bryll [1996].

5.3. Maduch's criterion. As we have seen, Łukasiewicz's rules are not always sufficient to achieve Ł-decidability of many logical systems. However, in the case of a standardly formalized sentential calculus, we can easily direct our search for a proper complete refutation system thanks to a criterion formulated by Maduch [1973]. Adapting the notions introduced in Sections 2-4 (asserted system, refutation system, complete refutation system, Ł-completeness and Ł-decidability) we define the criterion in the following way:

MADUCH'S THEOREM. *Let, for a given sentential logic, the system* $\langle F, A^+, R^+\rangle$, *where* $R^+ = \{r_1^+, r_2^+\}$, *be asserted. Then, if for this logic there exists a* basis directed downwards, *then there is no finite set* A^- *of rejected axioms such that* $\langle F, A^-, R^-\rangle$ *is a complete refutation system, i.e.* $T^- = F\setminus T^+$, *where* $T^- = Cn^-(T^+ \cup A^-, R^-)$, $R^- = \{r_1^-, r_2^-\}$ (see D(Ł)).

Let us note that the condition $T^- = F\setminus T^+$ is equivalent to the condition 2° of Ł-completeness and Ł-decidability because $T^+ = E(M)$ and $A^- \subseteq F\setminus E(M)$ for the adequate matrix M for this system of logic.

The *basis directed downwards* for this logic is a non-empty family $J \subseteq 2^F$ satisfying the following conditions:

1° $X, Y \in J \Rightarrow \exists Z \in J (Z \subseteq X \cap Y)$,

2° $\bigcap J = T^+$,

3° $\forall X \in J (T^+ \subset X \wedge X = Cn^+(X, R^+))$, where Cn^+ is a usual consequence operation.

On the basis of this criterion it is possible to prove that such systems as, for example, intuitionistic sentential calculus, $\aleph_0$-valued implicative-negative Łukasiewicz's logic, Lewis system S5, and others with Łukasiewicz's rules are not Ł-decidable.

5.4. The generalized method of natural deduction. Ending this review of the more important results of research into Ł-decidability of deductive system we must emphasize that they concerned one axiomatic method of rejection introduced to metalogical investigations by Łukasiewicz. In many cases, the method can be replaced with the generalized method of natural deduction. The latter is applicable to all propositional calculi which have finite adequate matrices, as well as to the intuitionistic propositional calculus and the first-order predicate calculus, that is, to almost all logics discussed in this section. The basis for such systems consists, then, of only assertion rules and rejection rules; the sets asserted and the rejected axioms are empty sets. The method used in the proofs is similar to Słupecki–Łukasiewicz's method of rejection though they are apagogic proofs (by *reductio ad absurdum*). This method refers to the 'tableaux' method. It is presented by Bryll [1996], who in his studies refers to results obtained by Hintikka [1955], Smullyan [1968], Suchoń [1977], Surma [1974a, b] and Carnielli [1987], [1991].

6. ŁUKASIEWICZ'S FUNCTION

The notion of rejected proposition, introduced to logic by Łukasiewicz (see D(Ł)), taken in Słupecki's equivalent approach (closer to Aristotle's idea (see D(Sł))), has been the subject of systematic, metalogical investigations initiated by Słupecki. The biaspectual characterization of deductive systems, introduced by Łukasiewicz, permits, on one hand, to mean the set T^+ as a set of accepted consequences (inferences), and, on the other hand, the set T^- as a set of rejected consequences (inferences). The syntactic assertion of a proposition (a formula) consists in deriving it from an earlier asserted one, while the syntactic rejection of a proposition consists in deriving from it a proposition that has already been rejected (cf. D(Sł) and D_S(Sł)). Properties of deductive systems based on asserted systems and the notion of derivability (assertion consequence) are established, as is known, by the general theory of deductive systems, i.e. the axiomatic theory of consequence built by Alfred Tarski [1930a]. Now, a question arises whether it is possible to extend this theory in such a way that a resulting theory would characterize deductive systems determined by their whole bases, that is, also by refutation systems and the notion of rejection (rejection consequence).

A positive answer to this question was given by Słupecki [1959] and his disciple: Wybraniec-Skardowska [1969] (cf. also Bryll [1969]). As was noticed by Słupecki, the notion of rejected proposition is so general that it is most convenient to base studies concerning this notion on Tarski's theory of deductive systems. Let us recall that the only primitive notions of this theory are: the set F of all propositions (well-formed formulas) of an arbitrary but fixed language of a given system and the consequence operation Cn^+: $2^F \to 2^F$; the symbol '$Cn^+(X)$' denotes the set of all consequences of the set $X \subseteq F$. Properties of deductive systems characterized by the bases with refutation systems can be established by means of Tarski's theory enriched by the following definition of rejected operation Cn^{-1}: $2^F \to 2^F$, determined by the consequence operation Cn^+ (see D(Sł)): For any $X \subseteq F$, $\alpha \in F$

MD(Sł). $\alpha \in Cn^{-1}(X) \Leftrightarrow \exists \beta \in X (\beta \in Cn^+(\{\alpha\}))$.

According to the definition MD(Sł): a proposition α is a *rejected proposition* on the ground of the set X if and only if at least one of the propositions of X is derivable from (is a consequence of) α; the symbol '$Cn^{-1}(X)$' denotes the set of all propositions rejected on the ground of propositions of X.

The following theorem helps to understand the intuitive sense of the definition MD(Sł):

$$\forall X \subseteq F\,(X \subseteq Y \Rightarrow Cn^{+}(X) \subseteq Y) \Rightarrow \forall X \subseteq F\,(X \subseteq Y' \Rightarrow Cn^{-1}(X) \subseteq Y').$$

If Y is the set of all true propositions of F (then the set Y' is the set of all false propositions of F), then, according to the above theorem, we can state that: If a consequence operation always leads from true propositions to true propositions, then a rejected operation always leads from false propositions to false propositions. Thus, the definition of the rejected operation corresponds to the notion of the rejection of proposition introduced to formal logic by Łukasiewicz. So, the defined operation (function) Cn^{-1} is a generalization of the notion of rejection introduced by Łukasiewicz.

The definition MD(Sł) of the function Cn^{-1} was formulated by Słupecki [1959], and was called by him *Łukasiewicz's function*. Słupecki proved that the function satisfies all axioms of Tarski's general theory of deductive systems for the consequence Cn^{+} and that it is additive. Thus, Łukasiewicz's function is another consequence operation and is called a *rejection consequence*. It is clear that every deductive system with a bi-level formalization (with the asserted and the refutation systems) can be characterized by the basis:

$$\langle F, Cn^{+}, Cn^{-1} \rangle$$

and that the extension of Tarski's theory of the definition MD(Sł) describes every such system. This theory has been developed in the form of the theory of rejected propositions by Wybraniec-Skardowska [1969], and later also by Bryll [1969]. Their researches have been a continuation of investigations initiated by Słupecki and have been conducted under his supervision to be later presented in co-authored papers (Słupecki, Bryll, Wybraniec-Skardowska [1971], [1972]).

The theory of rejected propositions has been based on the enriched theory of deductive systems built by Tarski in [1930b][3] and includes, apart from the definition MD(Sł) of Łukasiewicz's function (the rejection consequence) Cn^{-1}, all counterparts with respect to rejection of such notions of Tarski's theory as: a system, a set of axioms, a set of independent propositions, a consistent set, a complete set. The theory of rejected propositions contains many significant theorems that are not counterparts of any theorem of Tarski's theory. The following are examples of such theorems:

$$Cn^{-1}(\emptyset) = \emptyset,$$

$$Cn^{-1}(\bigcup \{X \subseteq F' \colon F' \subseteq F\}) = \bigcup \{Cn^{-1}(X) \colon X \subseteq F' \subseteq F\},$$

(*) $$Cn^{-1}(X) = \bigcup \{Cn^{-1}(\{\alpha\}) \colon \alpha \in X\},$$

(**) $$\alpha \in Cn^{-1}(X) \Leftrightarrow \exists \beta \in X\,(Cn^{-1}(\{\alpha\}) \subseteq Cn^{-1}(\{\beta\})).$$

The above theorems state that Łukasiewicz's function is an operation that is, respectively, normal, completely additive, unit and unit consequence. In the theory of rejected propositions the following theorem is valid, which is a formulation of the rule of rejection by detachment:

$$c\,\alpha\,\beta \in Cn^{+}(Y) \wedge \beta \in Cn^{-1}(Y \cup X) \Rightarrow \alpha \in Cn^{-1}(Y \cup X),$$

while in Tarski's theory the following theorem, being a formulation of the rule of detachment, is valid:

$$c\,\alpha\,\beta \in Cn^{+}(Y) \wedge \alpha \in Cn^{+}(Y \cup X) \Rightarrow \beta \in Cn^{+}(Y \cup X).$$

The set $Y \subseteq F$ can be here understood as a set A^{+} of axioms or a set T^{+} of theses of a given deductive system.

The relations between the consequence Cn^{+} and the rejection consequence Cn^{-1} are determined, particularly, by the following theorems (cf. Wybraniec-Skardowska [1969]):

$$\beta \in Cn^{+}(\{\alpha\}) \Leftrightarrow \alpha \in Cn^{-1}(\{\beta\}),$$

$$Cn^{+}(X) = NCn^{-1}(ANX \cup \{n\,c\,\alpha\,\alpha\}) = NCn^{-1}N(KX \cup \{c\,\alpha\,\alpha\}),$$

where, for any $Y \subseteq F$, NY is the set of all propositions contradictory to these belonging to the set Y, AY is the set of all alternatives which are built by means of propositions of the set Y, and KY is the set of all conjunctions of such propositions.

This last theorem points out the possibility of the axiomatization of a theory of rejected propositions in a dual and equivalent way, i.e. assuming that a primitive notion is the rejection consequence Cn^{-1}, while Cn^{+} is the defined operation. Such a dual theory of rejected propositions was formulated by Wybraniec-Skardowska [1969] (see also Słupecki, Bryll, Wybraniec-Skardowska [1971]). It can be understood as the theory describing the deductive systems with the basis

$$\langle F, Cn^{-1}, Cn^{+} \rangle.$$

A theory of rejected propositions was developed also to formalize some problems of methodology of empirical sciences, mainly by Bryll [1969] (see

[3] We should remember that this theory has two new terms: 'c' and 'n', denoting functors of implication and negation, respectively. The symbol '$Cn^{+}(X)$' denotes here a set of all consequences of X with respect to the set of classical logical axioms and the rule of detachment. So, $Cn^{+}(\emptyset)$ is the set of all substitutions of theses of classical logic.

also Słupecki, Bryll, Wybraniec-Skardowska [1972]). The theories of rejected propositions have a natural interpretation, which was given by Staszek [1973]: the set $Cn^{-1}(X)$ can be understood as a set of all rejection proofs on the ground of a proposition of the set X, in the sense relating to the nature of rejection proofs used by Słupecki in his researches on Aristotle's syllogistic.

In the theory of rejected proposition, Słupecki defined the notions of Ł-consistence, Ł-completeness and Ł-decidability (see Słupecki, Bryll, Wybraniec-Skardowska [1971]): For any $X, Y \subseteq F$

$$\langle X, Y\rangle \in \text{Ł-consistent} \Leftrightarrow Cn^{+}(X) \cap Cn^{-1}(Y) = \emptyset,$$

$$\langle X, Y\rangle \in \text{Ł-complete} \Leftrightarrow Cn^{+}(X) \cup Cn^{-1}(Y) = \emptyset,$$

$$\langle X, Y\rangle \in \text{Ł-decidable} \Leftrightarrow \langle X, Y\rangle \in \text{Ł-consistent} \cap \text{Ł-complete}.$$

Łukasiewicz's function can be generalized into a dual, finitistic consequence in the usual meaning. The notion of the dual consequence dCn^{+} relating to the consequence Cn^{+} was introduced by Wójcicki [1973] and has the following definition:

$DdCn^{+}$.

$$\alpha \in dCn^{+}(X) \Leftrightarrow \exists Y \subseteq X \wedge \operatorname{card}(Y) < \aleph_0 \left(\bigcap \{Cn^{+}(\{\beta\}) \colon \beta \in Y\} \subseteq Cn^{+}(\{\alpha\})\right).$$

The dual consequence dCn^{+} is stronger than the rejected consequence Cn^{-1} (i.e. $Cn^{-1} \leqslant dCn^{+}$) though the former is linked with the latter by a number of interesting relationships (see Spasowski [1973]).

The rejection consequence Cn^{-1} is a unit operation (in the sense of (*)), and hence a unit consequence (see (**), cf. Wybraniec-Skardowska [1969]). It is a consequence that is *mutually dual* with the unit consequence Cn^{-1} defined as follows (see Słupecki, Bryll, Wybraniec-Skardowska [1970]):

DCn^{+1}. $\quad \alpha \in Cn^{+1}(X) \Leftrightarrow \exists \beta \in X\, (\alpha \in Cn^{+}(\{\beta\}))$.

It is clear that (cf. (**))

$$\alpha \in Cn^{+1}(X) \Leftrightarrow \exists \beta \in X\, (Cn^{+1}(\{\alpha\}) \subseteq Cn^{+1}(\{\beta\})).$$

Dual consequences relating to the consequences Cn^{+1} and Cn^{-1} can be denoted by dCn^{+1} and dCn^{-1}, respectively. We define them putting in the definitions DCn^{+1} and MD(Sł) of consequences Cn^{+1} and Cn^{-1} the term 'dCn^{+}' instead of the term 'Cn^{+}'. Then from the definitions $DdCn^{+}$, MD(Sł) and DCn^{+1} we obtain

FACTS. a. $dCn^{+1} = Cn^{-1}$ *and* b. $dCn^{-1} = Cn^{+1}$.

The Facts state that the consequences Cn^{+1} and Cn^{-1} are *mutually dual*.

The consequence Cn^{+} (Cn^{+1}) remains in a close relationship with the consequence defined semantically, when a deductive system has an adequate matrix. The dual consequence dCn^{+} (dCn^{+1}) made it possible to study the so-called dual logics in relation to Łukasiewicz's logics (stated either by means of a consequence defined by matrix or by means of a matrix consequence (cf. Malinowski, Spasowski [1974])).

The dual consequences Cn^{+} and dCn^{+} as well as Cn^{+1} and Cn^{-1} can be used to study both true (asserted) and, respectively, false (rejected) contents of a given theory. The fact was noticed by Woleński [1989] in his studies relating to Popper's conception of a comparison of scientific theories by their contents.

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